LESSON

16 Areas under Curves

Around the same time that so many great mathematicians devoted themselves to figuring out the slopes of tangent lines, other mathematicians were working on an entirely different problem. They wanted to be able to figure out the area underneath any curve $y = f(x)$, such as the one shown in Figure 16.1.

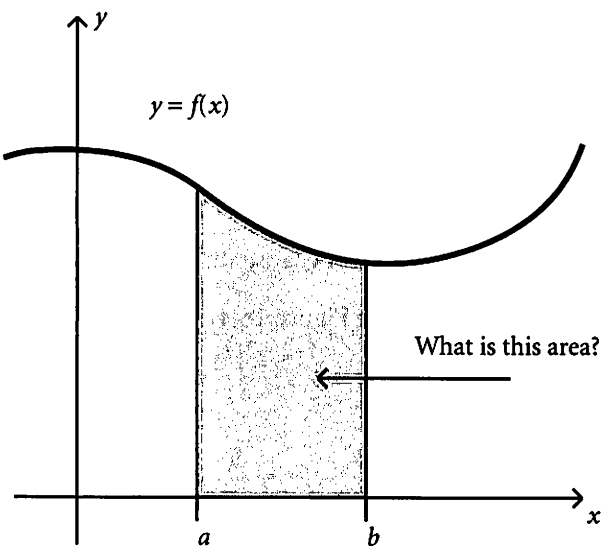


Figure 16.1

They used the shorthand notation $\int_a^b f(x)\,dx$ to represent the area between, or bound by, the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$. This symbol is generally referred to as an *integral*. Note: The dx is merely to indicate which letter is the variable. If the horizontal axis represented time, for instance, then this would end in dt.

Example

Evaluate the integral $\int_0^4 \frac{1}{2}x\,dx$.

Solution

This represents the area between the curve $y = \frac{1}{2}x$, the x-axis, the line $x = 0$, and the line $x = 4$ (see Figure 16.2). This area happens to be a triangle with a height of 2 and a base of 4. The area of the triangle is $\frac{1}{2}(2)(4) = 4$. Thus, $\int_0^4 \frac{1}{2}x\,dx = 4$.

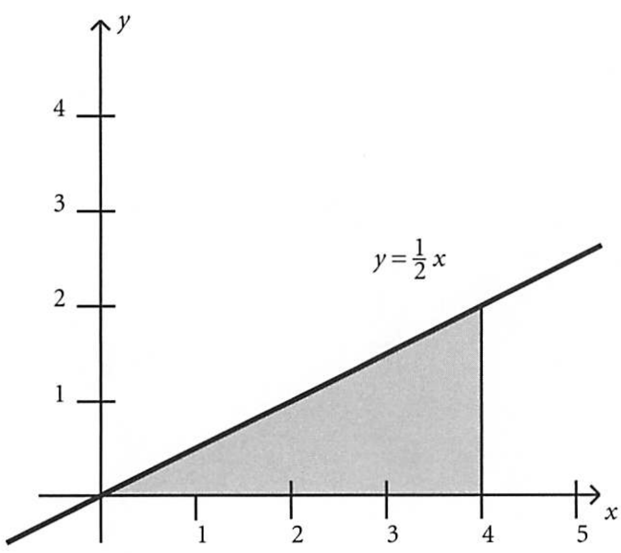


Figure 16.2

Also, just to be sporting, area *below* the x-axis is counted negatively. Therefore, really $\int_a^b f(x)\,dx$ represents "the area between the curve $y = f(x)$, the x-axis, $x = a$, and $x = b$, where area below the x-axis is counted negatively."

Example

Evaluate the integrals $\int_1^3 f(x)\,dx$, $\int_1^4 f(x)\,dx$, $\int_1^6 f(x)\,dx$, and $\int_6^7 f(x)\,dx$ where the graph of $y = f(x)$ is given as shown in Figure 16.3.

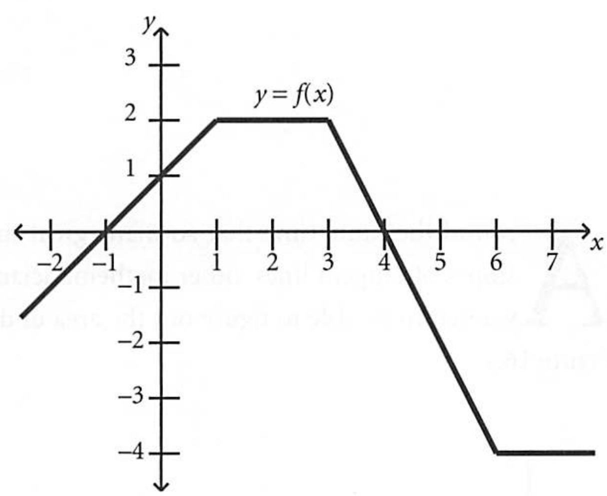


Figure 16.3

Solution

First, $\int_1^3 f(x)\,dx = 4$ because this area is a square above the x-axis (see Figure 16.4).

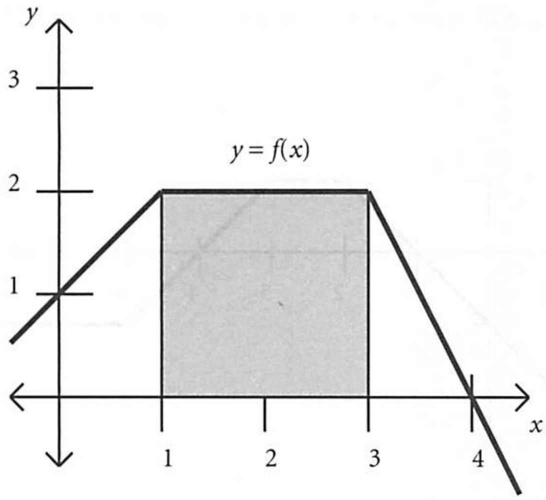


Figure 16.4

Next, $\int_1^4 f(x)\,dx = 4 + \frac{1}{2}\cdot(1)\cdot 2 = 5$ because this area is a square plus a triangle (see Figure 16.5).

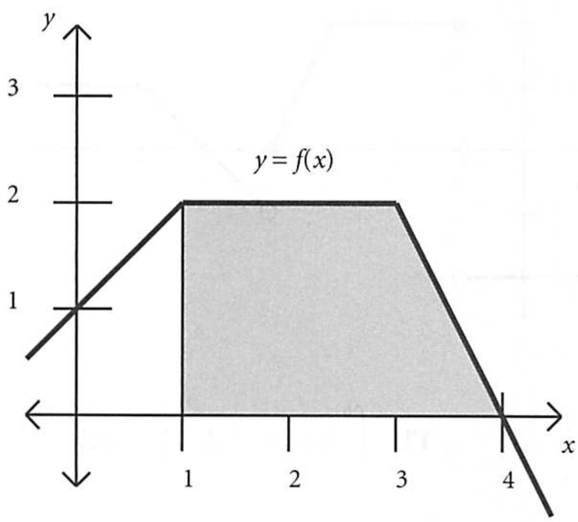


Figure 16.5

For $\int_1^6 f(x)\,dx$, we must calculate how much area is above the x-axis and how much is below (see Figure 16.6).

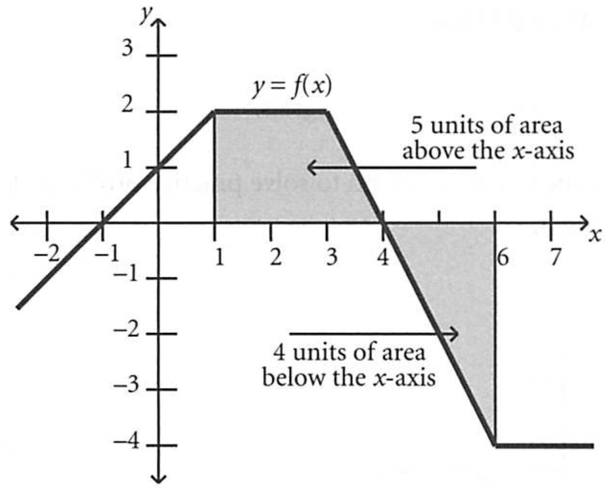


Figure 16.6

There are 5 units of area above the x-axis and 4 units below, so $\int_1^6 f(x)\,dx = 5 - 4 = 1$.

Finally, $\int_6^7 f(x)\,dx$ represents a rectangle of area 4 that is entirely below the x-axis. Thus, $\int_6^7 f(x)\,dx = -4$ (see Figure 16.7).

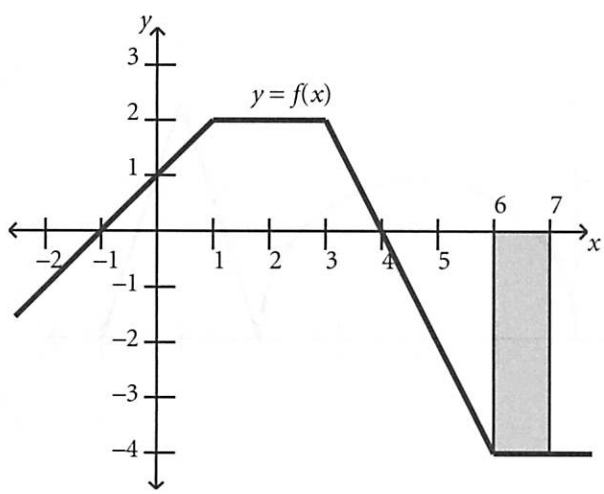
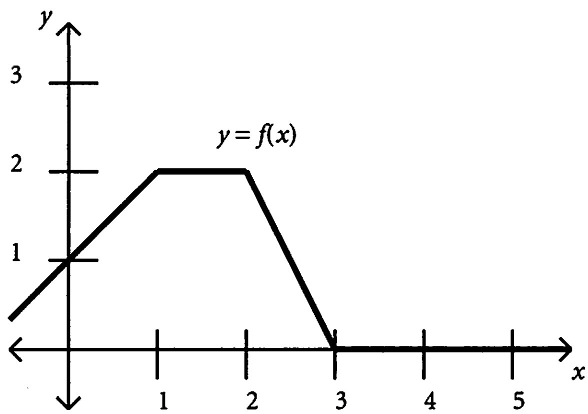


Figure 16.7

▶ Practice

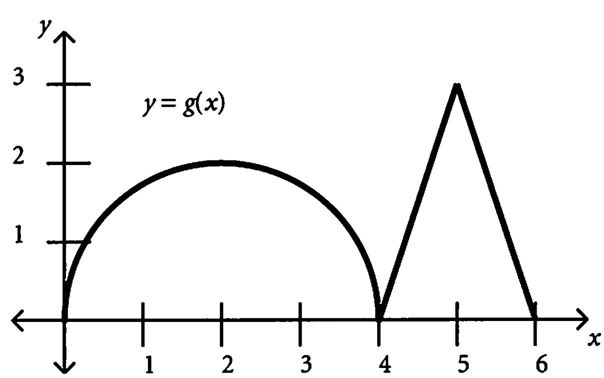
Evaluate the following integrals.

Use the following graph to solve practice problems 1, 2, and 3.



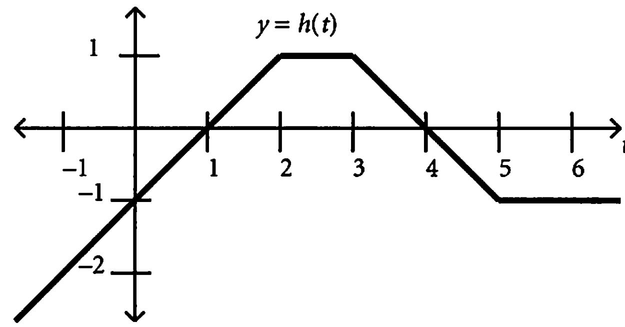
1. $\int_1^2 f(x)dx$ **2.** $\int_0^1 f(x)dx$ **3.** $\int_0^2 f(x)dx$

Use the following graph for practice problems 4, 5, and 6.



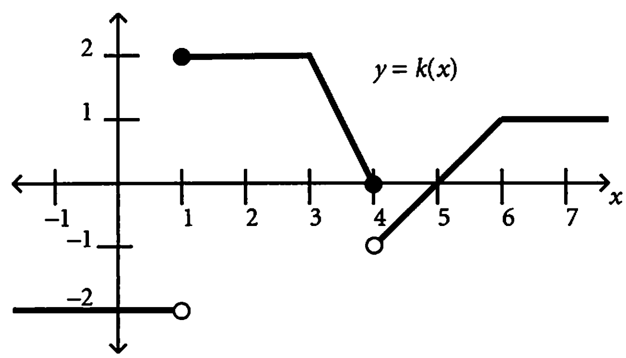
4. $\int_0^4 g(x)dx$ **5.** $\int_4^6 g(x)dx$ **6.** $\int_0^6 g(x)dx$

Use the following graph for practice problems 7, 8, and 9.



7. $\int_{-1}^6 h(t)dt$ **8.** $\int_{-1}^4 h(t)dt$ **9.** $\int_4^6 h(t)dt$

Use the following graph for practice problems 10 through 18.



10. $\int_0^7 k(x)dx$ **11.** $\int_4^6 k(x)dx$ **12.** $\int_4^5 k(x)dx$

13. $\int_0^4 (x+2)dx$ **14.** $\int_1^4 2dx$ **15.** $\int_1^5 (t-3)dt$

16. $\int_{-2}^5 xdx$ **17.** $\int_1^6 2xdx$ **18.** $\int_0^8 (2x-2)dx$

You may have noticed:

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

The area between a and c is the area from a to b plus the area from b to c, assuming, of course, that $a < b < c$ (see Figure 16.8).

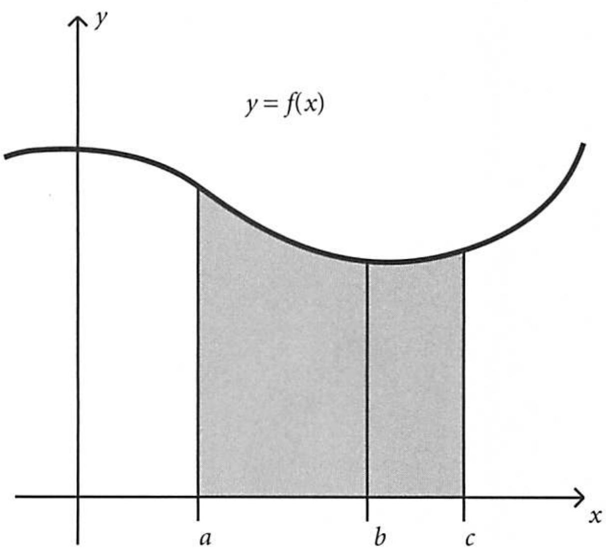


Figure 16.8

Similarly, $\int_a^c f(x)\,dx - \int_b^c f(x)\,dx = \int_a^b f(x)\,dx$.

We can use these to make calculations, even when the exact functions are unknown.

Example

If $\int_3^5 f(x)\,dx = 7$ and $\int_5^{10} f(x)\,dx = 15$, then what is $\int_3^{10} f(x)\,dx$?

Solution

$$\int_3^{10} f(x)\,dx = \int_3^5 f(x)\,dx + \int_5^{10} f(x)\,dx$$
$$= 7 + 15 = 22$$

Example

If $\int_0^{10} g(x)\,dx = 38$ and $\int_8^{10} g(x)\,dx = -12$, then what is $\int_0^8 g(x)\,dx$?

Solution

$$\int_0^8 g(x)\,dx = \int_0^{10} g(x)\,dx - \int_8^{10} g(x)\,dx$$
$$= 38 - (-12) = 50$$

▶ Practice

Suppose $\int_0^6 f(x)\,dx = 10$, $\int_6^7 f(x)\,dx = -5$, and $\int_7^{11} f(x)\,dx = 2$. Evaluate the following.

19. $\int_0^7 f(x)\,dx$ **20.** $\int_6^{11} f(x)\,dx$ **21.** $\int_0^{11} f(x)\,dx$

Suppose $\int_1^{14} g(t)\,dt = -3$, $\int_{10}^{14} g(t)\,dt = 8$, and $\int_1^5 g(t)\,dt = -10$. Evaluate the following.

22. $\int_5^{14} g(t)\,dt$ **23.** $\int_1^{10} g(t)\,dt$ **24.** $\int_5^{10} g(t)\,dt$

Suppose $\int_{-2}^{11} h(x)\,dx = 20$, $\int_{-2}^{1} h(x)\,dx = 12$, and $\int_{-2}^{10} h(x)\,dx = -5$. Evaluate the following.

25. $\int_1^{11} h(x)\,dx$ **26.** $\int_1^{10} h(x)\,dx$ **27.** $\int_{10}^{11} h(x)\,dx$